Local bisection for conformal refinement of unstructured 4D simplicial meshes

Guillem Belda-Ferrín*, Abel Gargallo-Peiró*, Xevi Roca*
*Barcelona Supercomputing Center (BSC), Barcelona, Spain
E-mail: {guillem.belda, abel.gargallo, xevi.roca}@bsc.es

***Abstract*—We present a conformal bisection procedure for local refinement of 4D unstructured simplicial meshes with bounded minimum shape quality. Specifically, we propose a recursive refine-to-conformity procedure in two stages, based on marking bisection edges on different priority levels and defining specific refinement templates. Two successive applications of the first stage ensure that any 4D unstructured mesh can be conformingly refined. In the second stage, the successive refinements lead to a cycle in the number of generated similarity classes and thus, we can ensure a bound over the minimum shape quality. In the examples, we check that after successive refinement the mesh quality does not degenerate. Moreover, we refine a 4D unstructured mesh and a space-time mesh (3D + 1D) representation of a moving object.**

***Keywords*—Meshing, Adaptative, 4D.**

I. Extended Abstract

A. Introduction

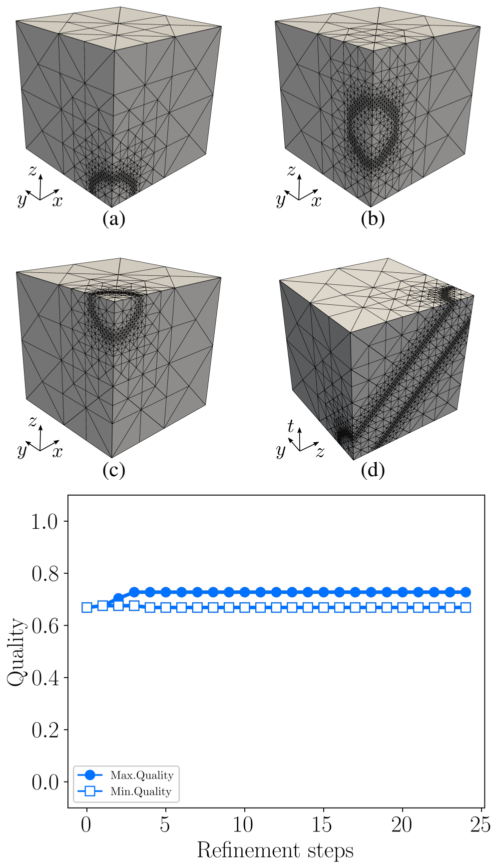
In the last three decades refinement of 2D and 3D unstructured simplicial meshes [1]–[14], based on red/green refinement [1]–[7] and bisection [8]–[14], has been shown to be a key ingredient on efficient adaptive loops. Although one could expect the same in 4D, a case of special interest for space-time adaption, this line of research has not been extensively explored. For our space-time applications, we are interested in conformal bisection methods since they are really well suited to implement fast geometrical multi-grid conformal solvers. Moreover, bisection methods have ensured either a maximum number of generated similarity classes [11]–[13] or a minimum lower quality bound over the generated elements after successive refinements [8]–[10], [14]. Regarding 4D refinement, only a non-conformal local refinement method for pentatopic meshes has been proposed [15]. Unfortunately, existent conformal 4D (nD) bisection methods with a bound over the number of generated similarity classes [11], [12] cannot be applied to general unstructured meshes.

The main contribution of this work is to propose a local bisection procedure, with a bound over the number of generated similarity classes, for conformal refinement of 4D unstructured simplicial meshes. Specifically, we propose a recursive refine-to-conformity procedure, in two stages, based on marking bisection edges on different priority levels. The marking procedure allows classifying the pentatopes in different types and hence, determining different refinement templates, in an analogous manner to the 3D bisection method proposed in [13]. The refinement method is composed of two stages. Two successive applications of the initial stage of the bisection strategy , based on the proposed element classification, ensure that any initial 4D unstructured simplicial mesh can be conformingly refined. After the two initial refinements our recursive refine-to-conformity strategy switches to the second stage. This final stage is analogous to Maubach's algorithm, when it is successively applied to a single pentatope. Therefore, we can ensure a bound over the number of generated similarity classes. Thus, the minimum quality of the refined mesh is bounded, independently of the number of performed refinements. The main advantage and difference of our method when compared to Maubach's algorithm [11] is the first stage of the method, which allows the application of the method to any 4D unstructured simplicial mesh.

In the presented example we show that the proposed methodology leads to a periodic evolution of the minimum element quality illustrating the lower bound of the quality through successive refinement. We first illustrate how to check that an implementation of the proposed method is valid by successively refining a pentatope. With our implementation, we show that the proposed bisection technique can be used to refine general unstructured 4D meshes. Finally, we also illustrate our application of interest, the refinement of a 4D mesh corresponding to a space-time representation, with varying resolution, of the temporal evolution of a 3D moving object.

B. Examples

We illustrate our application of interest, the refinement of a 4D mesh corresponding to a space-time representation, with varying resolution, of the temporal evolution of a 3D moving object, see Figure 1. We generate an initial mesh on the hypercube $[0,1]^4$ composed by 24 pentatopes using Freudenthal-Kuhn algorithm [1]–[3]. Next, we apply 25 times the algorithm `RefineToConformity` to refine those elements that intersect the 4D sphere extrusion that represents the moving sphere. The final 4D mesh is composed by 5233296 pentatopes and 251457 nodes and it is illustrated in Figure 1. Figures 1(a)-1(c) show three slices of the mesh at $t = 0$, $t = 1/2$ and $t = 1$, respectively. We can observe that each one of the slices on t shows different positions of the moving sphere, from the initial point (0,0,0) at $t = 0$ to the final point $(0, 0, 1)$ at $t = 1$. In contrast with these three slices, in Figure 1(d) we show an slice of the mesh at $x = 0$. In the closest quadrilateral face of Fig. 1(d) we observe the path of the sphere on the surface of dimension 2 defined by the axis z and t at $x = y = 0$. In this quadrilateral face, we can see that the center of the sphere describes a straight line going from the lower left corner $(0, 0, 0, 0)$ up to the top right corner $(0, 0, 1, 1)$. This is so since the sphere goes from $z = 0$ to $z = 1$ with constant velocity starting at $t = 0$ and finalizing at $t = 1$. Specifically, the location on the z-axis of the sphere is $z = t$.



(e)

Figure 1. Slice of the 4D simplicial mesh of the hypercube with the hyperplane: (a) $t = 0$, (b) $t = 0.5$, (c) $t = 1$ and (d) $x = 0$. (e) Minimum (blue) and maximum (red) element quality for each refinement step.

C. Conclusion

In this work, we have presented a new refinement method via edge bisection for 4D pentatopic meshes. This method ensures that the mesh quality does not degenerate after successive refinements of a given element. To develop this method, we require to classify the elements of the mesh into different types in a similar fashion to [13]. Using the pentatope classification we provide four refinement templates to perform a cyclic bisection analogous to Maubach's method [11]. Combining two initializing refinements (Stage 1) with this templated refinement (Stage 2) we obtain a refinement strategy that can be applied to any given pentatopic mesh. Using this method a finite number of similarity classes are generated when a given element is refined. We apply the refinement scheme to different meshes to illustrate its features. First, we analyze that the mesh quality of the refinement of different element types does not degenerate. Second, we illustrate the applicability of the technique to refine unstructured 4D simplicial meshes. Finally, we analyze a space-time configuration of a sphere moving along an axis.

II. Acknowledgment

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 715546. This work has also received funding from the Generalitat de Catalunya under grant number 2017 SGR 1731. The work of X. Roca has been partially supported by the Spanish Ministerio de Economía y Competitividad under the personal grant agreement RYC-2015-01633..

References

[1] H. Freudenthal, "Simplizialzerlegungen von beschrankter flachheit," *Ann. Math.*, pp. 580–582, 1942.

[2] H. Kuhn, "Some combinatorial lemmas in topology," *IBM Journal of research and development*, vol. 4, no. 5, pp. 518–524, 1960.

[3] J. Bey, "Simplicial grid refinement: on freudenthal's algorithm and the optimal number of congruence classes," *Numerische Mathematik*, vol. 85, no. 1, pp. 1–29, 2000.

[4] R. Bank *et al.*, "Some refinement algorithms and data structures for regular local mesh refinement," *Scientific Computing*, vol. 1, pp. 3–17, 1983.

[5] J. Bey, "Tetrahedral grid refinement," *Computing*, vol. 55, no. 4, pp. 355–378, 1995.

[6] A. Liu and B. Joe, "Quality local refinement of tetrahedral meshes based on 8-subtetrahedron subdivision," *Mathematics of Computation*, vol. 65, no. 215, pp. 1183–1200, 1996.

[7] S. Zhang, "Successive subdivisions of tetrahedra and multigrid methods on tetrahedral meshes," *Houston J. Math*, vol. 21, no. 3, pp. 541–556, 1995.

[8] M. Rivara, "Algorithms for refining triangular grids suitable for adaptive and multigrid techniques," *International journal for numerical methods in Engineering*, vol. 20, no. 4, pp. 745–756, 1984.

[9] E. Bänsch, "Local mesh refinement in 2 and 3 dimensions," *IMPACT of Computing in Science and Engineering*, vol. 3, no. 3, pp. 181–191, 1991.

[10] A. Liu and B. Joe, "Quality local refinement of tetrahedral meshes based on bisection," *SIAM Journal on Scientific Computing*, vol. 16, no. 6, pp. 1269–1291, 1995.

[11] J. Maubach, "Local bisection refinement for n-simplicial grids generated by reflection," *SIAM Journal on Scientific Computing*, vol. 16, no. 1, pp. 210–227, 1995.

[12] C. Traxler, "An algorithm for adaptive mesh refinement in n dimensions," *Computing*, vol. 59, no. 2, pp. 115–137, 1997.

[13] D. Arnold *et al.*, "Locally adapted tetrahedral meshes using bisection," *SIAM Journal on Scientific Computing*, vol. 22, no. 2, pp. 431–448, 2000.

[14] A. Plaza and M. Rivara, "Mesh refinement based on the 8-tetrahedra longest-edge partition." in *IMR*, 2003, pp. 67–78.

[15] M. Neumüller and O. Steinbach, "A flexible space-time discontinuous galerkin method for parabolic initial boundary value problems," *Berichte aus dem Institut für Numerische Mathematik*, vol. 2, 2011.

Guillem Belda Ferrín received his BSc degree in Mathematics from Universitat Politècnica de Catalunya, Barcelona in 2014. He completed his MSc degree in Advanced Mathematics and Mathematical Engineering from Universitat Politècnica de Catalunya in 2015. Since 2017, he has been with the Geometry and Meshing for simulations group of Barcelona Supercomputing Center (BSC) as a PhD student of the Applied Mathematics doctorate program of Universitat Politècnica de Catalunya (UPC), Spain.